SPACE CHARGE EFFECTS IN FFAG *

M. Haj Tahar, F. Méot , BNL, Upton, Long Island, New York, USA

Abstract

Understanding space charge effects in FFAG is crucial in order to assess their potential for high power applications. This paper shows that, to carry out parametric studies of these effects in FFAG, the average field index of the focusing and defocusing magnets are the natural parametrization. Using several classes of particle distribution functions, we investigate the effects of space charge forces on the nonlinear beam dynamics and provide stability diagrams for an FFAG-like lattice. The method developed in this study is mainly applicable to systems with slowly varying parameters, i.e slow acceleration.

INTRODUCTION

Since the early 1990s, cyclotrons and linacs have been proposed to address the mission of coupling a high power proton accelerator with a spallation target in an Accelerator Driven Subcritical Reactor (ADSR) application. However, due to the recent revival of interest in using FFAG accelerators, the question then arises as to what sort of improvement can the FFAG technology bring. In an effort to demonstrate the high beam power capability of FFAG, studies of the collective effects such as the space charge effects have been undertaken and plans for high intensity experiments are ongoing [1]. Benchmarking work has also been undertaken to validate the simulation results. The following study is in major part induced by the successful benchmarking work so far [2].

AVERAGE FIELD INDEX OF THE FOCUSING/DEFOCUSING MAGNET

In a FFAG accelerator, it is not possible to define a single closed orbit. Instead, the beam moves outwards, and therefore, all parameters are likely to change with the energy of the reference particle. For that reason, it is particularly misleading to replace the equation of motion with a transfer matrix since the betatron amplitude functions continuously change with the energy. For instance, in a scaling FFAG where the tunes are to remain constant, $\beta_{x,y} \propto R$ where R is the radius of the closed orbit [3].

In order to carry out parametric study of space charge effects in non-linear FFAG lattices, the idea is to vary the strength of the applied forces on the beam. For that reason, an extension of the mean field index k as defined by Symon [4], is to introduce its azimuthal variation by defining the mean field index of the focusing (F), defocusing (D) magnets and the drift ($drift$) in the following way:

$$k_i = \frac{R}{B_i}\frac{dB_i}{dR} \quad ; \quad i = F, D, drift \qquad (1)$$

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where B_i is the vertical component of the magnetic field in the median plane of the FFAG that is averaged over the width of the element. Since the drift space between the magnets is likely to contain the fringe fields, it may be important to assign a mean field index to it to determine its effect on the beam dynamics. However, in the ideal case, $k_{drift} = 0$, which we will assume in the following for simplification, yet without loss of generality. Now, assuming that the k values have no radial dependence, Eq. (1) can be integrated and the magnetic field expressed in cylindrical coordinates:

$$\begin{aligned} B(R,\theta) = \quad & B_{F0} \times \left(\frac{R}{R_0}\right)^{k_F} \times F_F(\theta) \\ + \quad & B_{D0} \times \left(\frac{R}{R_0}\right)^{k_D} \times F_D(\theta) \end{aligned} \qquad (2)$$

where F_F and F_D are the fringe field factors that describe the azimuthal variation of the field in the F and D magnets respectively. It is important to note that the field is a separable function in radial and azimuthal coordinates since the fringe fields merge to zero in the drift space between the magnets as can be seen in Fig. 1 below. Also, note that if $k_F = k_D$, the field writes in the standard form of a scaling FFAG. The lattice considered for this study is a radial sector KURRI-like DFD triplet [1]. Now assuming that all orbits

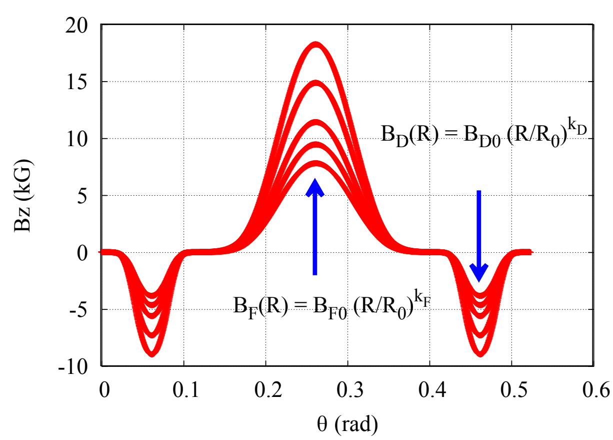

Figure 1: Magnetic field along several closed orbits.

are similar, i.e assuming a linear motion around the closed orbits in which only the path length and the field index may change with the radius, it can be shown that the square number of betatron oscillations in the transverse plane is a linear combination of the k_i values [3]. However, this result is not valid in the general case as shown in [3]: by using the Bogoliubov-Krilov-Mitropolsky (BKM)'s method of averages [5], one can compute approximately the frequencies of the betatron oscillations and their dependence on the average field index of the F and D magnets. To the first order

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in the BKM's method, this yields:

$$\nu_x^2(R_E) = \sum_i \beta_i(R_E) - \sum_i \alpha_i(R_E) \times k_i(R_E) \quad (3)$$

$$\nu_y^2(R_E) = \sum_i \alpha_i(R_E) \times k_i(R_E) + \mathcal{F}^2 \quad (4)$$

where $\mathcal{F}$ is the magnetic flutter defined by

$$\mathcal{F}^2 = \frac{\left\langle B^2 \right\rangle - \langle B \rangle^2}{\langle B \rangle^2} = \frac{\left\langle F^2 \right\rangle - \langle F \rangle^2}{\langle F \rangle^2} \quad (5)$$

α_i and β_i are defined as the 1^{st} and 2^{nd} order index of similarity of the closed orbits:

$$\alpha_i(R_E) = \frac{-1}{2\pi/N} \int_{\theta_i} \mu(R,\theta) d\theta = \frac{-1}{2\pi/N} \int_{\theta_i} \frac{R}{\rho} d\theta \quad (6)$$

$$\beta_i(R_E) = \frac{1}{2\pi/N} \int_{\theta_i} \mu(R,\theta)^2 d\theta = \frac{1}{2\pi/N} \int_{\theta_i} \left(\frac{R}{\rho}\right)^2 d\theta \quad (7)$$

and R_E refers to the radius of the closed orbit of energy E. This study shows that perturbing the k_i values of the magnets creates a closed orbit change such that the indices of similarity of the orbits as well as the magnetic flutter become energy dependent. The tunes follow as well and two regimes can be distinguished:

- If $k_D < k_F$ then the phase advance per cell is a decreasing function of the energy in both planes.
- If $k_D > k_F$ then the phase advance per cell is an increasing function of the energy in both planes.

In order to better understand these results, we define two new quantities in the calculation, the average as well as the rms values of the tunes over the closed orbits to quantify the average focusing strength of the applied forces on the beam and for any tune variation:

$$\nu_{x,y}^m = \left\langle \nu_{x,y} \right\rangle = \frac{1}{NCO} \sum_{i=1}^{NCO} \nu_{x,y,i} \quad (8)$$

$$\nu_{x,y}^{rms} = \left\langle \nu_{x,y}^2 \right\rangle^{1/2} = \left(\frac{1}{NCO} \sum_{i=1}^{NCO} \left(\nu_{x,y,i} - \nu_{x,y}^m \right)^2 \right)^{1/2} \quad (9)$$

We also define $\kappa = k_D - k_F$, the difference of the average field index of the F and D magnet. Particle tracking in ZGOUBI [6] is performed in order to solve the non-linear equation of motion using truncated Taylor expansions of the field and its derivatives up to the 5^{th} order: the closed orbits are first obtained, then the phase advance per cell is computed for each. A scan on k_F and k_D provides the stability diagrams in Fig. 2 (in order to remain within the allotted length of the paper, we only show the results in the vertical plane): qualitatively, in the case where $k_F = k_D = k$, the results are in good agreement with the Symon formula ($\nu_y^2 \approx -k$): increasing k increases the horizontal tune and decreases the vertical one. Besides, the RMS tune exhibits the expected behavior in the vicinity of the line $k_F = k_D$ where it becomes negligible. One can also observe that for large k values, increasing κ makes the orbits quickly unstable, thus the stability diagram shrinks. This result is expected since the second order term in the field expansion becomes non negligible for large k so that the system is no longer slowly (linearly) responding to perturbations:

$$B(R) = B_0 \times (\frac{R}{R_0})^k = B_0 \times (1 + \frac{x}{R_0})^k \quad ; \quad x \ll R_0$$
$$= B_0 \times \left[1 + k\frac{x}{R_0} + k(k-1)(\frac{x}{R_0})^2 + O\left((\frac{x}{R_0})^3\right) \right]$$

However, in the vicinity of the central line, i.e $\kappa = 0$, one key finding is that the RMS tunes are proportional to κ:

$$\nu_{x,y}^{rms} \approx a_{x,y}|k_F - k_D| = a_{x,y}|\kappa| \quad (10)$$

This result, combined with the previous observation that the tunes follow two regimes depending on whether $\kappa > 0$ or < 0 is particularly useful as it demonstrates that alternating κ does allow to obtain a non-scaling fixed tune FFAG. This proves that the two cardinal conditions of a scaling FFAG (similarity of the orbits and constancy of the field index with respect to the momentum) are sufficient but non necessary conditions to obtain a fixed tune FFAG. However, this result will be further discussed in a later publication.

SPACE CHARGE EFFECTS IN FFAG

In order to investigate the effects of space charge forces on the non-linear beam dynamics of FFAGs, the idea is to vary the (k_F, k_D) values which act upon the phase advance per cell in both planes as described in Eq. (3) and (4). This approach is in particular relevant to scaling FFAG: since it is impossible to make a field which corresponds exactly to the ideal one, i.e $k_F = k_D$, it is important to investigate the beam stability due to field errors in presence of space charge forces.

Procedure

- We build the model by generating a median plane field map for a given (k_F, k_D) as illustrated in Fig.1. Tracking is performed using ZGOUBI: Median plane antisymmetry is assumed and the Maxwell equations are accommodated which yields the Taylor expansions for the three components of the magnetic field.
- Search for NCO closed orbits between injection and extraction using the built-in fitting routines in ZGOUBI. NCO was chosen to be 30 in order to have good statistics and ensure the convergence of the calculated quantities. The energy range of the protons is chosen between 11 MeV and 100 MeV suitable for the KURRI 150 MeV FFAG.
- For each closed orbit, the matching condition ensuring a periodic motion around the closed orbit is obtained

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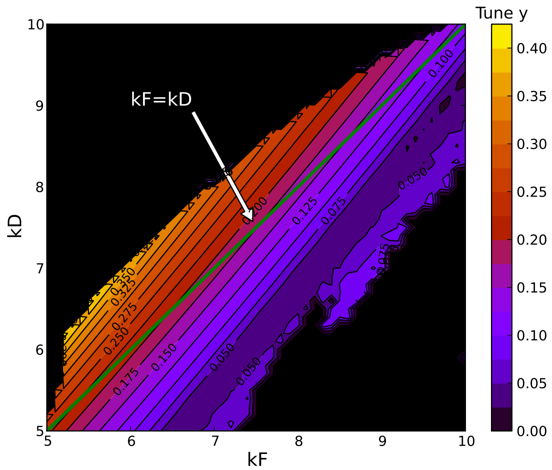

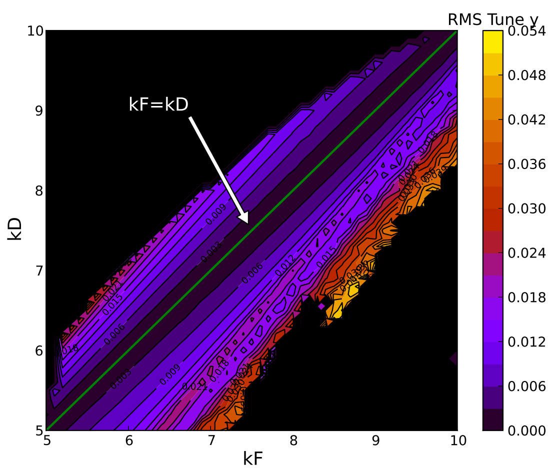

(b) RMS tune variations.

Figure 2: Contour plot of the average (a) and rms (b) tune variations in the vertical plane as a function of the scaling factors k_F and k_D: as can be observed from (b), the rms tune variations increase with increasing $|\kappa|$.

and the matched distribution is generated and tracked over several turns (120 turns). One assumes a symmetric beam distribution, i.e, $\epsilon_x = \epsilon_y = \epsilon$. The damping of beam emittances with momentum is taken into account.

- For each closed orbit, the number of betatron oscillations of the distribution is computed using the ZGOUBI Discrete Fourier Transform (DFT):

$$\nu^{co}_{x,y} = \left\langle \nu_{x,y} \right\rangle = \frac{1}{Npar} \sum_{i=1}^{Npar} \nu_{x,y,i} \tag{11}$$

$$\sigma^{co}_{\nu_{x,y}} = \left(\frac{1}{Npar} \sum_{i=1}^{Npar} \left(\nu_{x,y,i} - \nu^{co}_{x,y} \right)^2 \right)^{1/2} \tag{12}$$

Eqs. (8) and (9) are then invoked in order to calculate the average tune of the lattice as well as rms tune over the closed orbits.

- The same steps are repeated in presence and in absence of the space charge effects: given that the emittance growth time is much shorter than the synchrotron period, our numerical simulation consists of a frozen longitudinal phase space. This provides the tune depression. We assume that the FFAG is operating in emittance-dominated regime. Therefore, the space charge is treated as a small perturbation.

The closed orbits formalism that we use for our analysis is mainly valid under the assumption that all orbits are slowly changing with time, i.e the acceleration rate is small enough that the damping of the betatron oscillations can be considered adiabatic. This is essentially the case for the KURRI FFAG in which the energy increase $\Delta E = 2keV/turn$ is extremely slow that the tunes are independent of the acceleration rate.

KV Beam Distribution

For a fast parametric study, a frozen space charge model is employed. One has to keep in mind though that the KV model does not include non-linear effects that increase the emittance. This test is particularly useful to validate the model built. As described in [7], each particle in the distribution experiences a linear space charge kick given by:

$$\Delta x' = \frac{2Q}{(r_x + r_y) r_x} x \Delta s \tag{13}$$

$$\Delta y' = \frac{2Q}{(r_x + r_y) r_y} y \Delta s \tag{14}$$

where Q is the generalized perveance (dimensionless) defined by:

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m_0 c^2 \beta^2 \gamma^3} = \frac{I}{I_0} \frac{2}{\beta^3 \gamma^3} \approx 6.45 \times 10^{-8} \frac{I[A]}{(\gamma^2 - 1)^{3/2}} \tag{15}$$

In order to calculate the space charge tune shift, we recall the Laslett tune shift formula applied to a KV beam:

$$\Delta\nu_y = \frac{1}{4\pi} \oint_0^C \beta_y(s) \frac{2Q}{r_y (r_x + r_y)} ds \tag{16}$$

After simplification, and recalling the symmetry of the beam, this yields :

$$\frac{\Delta\nu_y}{\nu_{y_0}} \approx \frac{1}{2\pi} \frac{Q}{\epsilon^2} \frac{r_y^3}{r_x + r_y} \approx \frac{RQ}{\epsilon} \frac{\nu_{y_0}^{-3/2}}{\nu_{x_0}^{-1/2} + \nu_{y_0}^{-1/2}} \tag{17}$$

where ν_{y_0} is the undepressed vertical betatron tune and R is the average radius of the particle orbit in the accelerator. Interchanging x and y gives the horiontal tune shift. Introducing the normalized emittance $\epsilon_n = \beta\gamma\epsilon$ yields the scaling $\Delta\nu_y \propto R/(\beta\gamma^2)$.

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Case of scaling FFAGs Assuming that $\kappa = 0$, thus the undepressed phase advance per cell is constant, one applies the procedure described above to a scaling FFAG case by varying the average field index k of the magnets as well as the linear charge density λ of the beam, thus the perveance Q. The results of tracking using the space charge module implemented in ZGOUBI are shown in Fig. 3. In order to interpret the results, we inject the approximated tune formula: $\nu_{x_0}^2 \approx k+1$ and $\nu_{y_0}^2 \approx -k+\mathcal{F}^2$ into Eq. (17). This yields:

$$\frac{\Delta\nu_y}{\nu_{y_0}} \approx \frac{RQ}{\epsilon}\frac{(-k+\mathcal{F}^2)^{-3/4}}{(k+1)^{-1/4}+(-k+\mathcal{F}^2)^{-1/4}} \approx f(k,Q) \quad (18)$$

and a similar result can be obtained for the horizontal plane. A contour plot of the previous formula (Fig. 4) shows good agreement with the tracking results: increasing the k value increases the horizontal focusing and decreases the vertical one. Thus the tune shift exhibits the opposite behavior. Besides, the tune shift is quasi linear with respect to the perveance term, which is a good indication that the beam matching is well ensured as well. Some differences are observed between the analytical formula and the tracking results, mainly in the vertical plane: this is due to our assumption that the magnetic flutter $\mathcal{F}$ is independent of the k-value of the lattice, which is only an approximation.

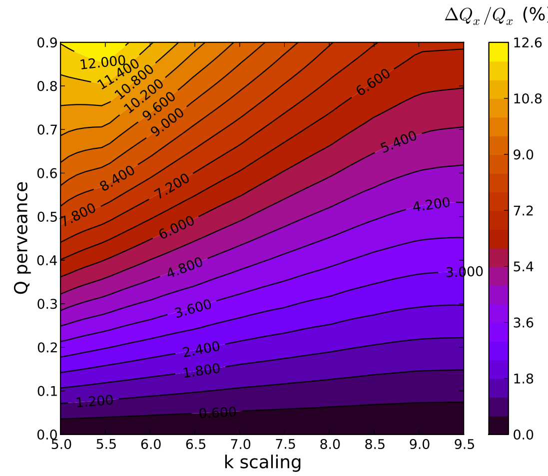
The above results hold for the average tune of the lattice. However, one shall recall that the tune excursion of a scaling FFAG in presence of space charge forces can be non negligible and scales as $R/(\beta\gamma^2)$. Therefore, one main question to answer is whether one can find a FFAG lattice that maintains the fixed tune property in presence of space charge. For that reason, the non-scaling FFAG case is treated next.

Case of non-scaling FFAGs In the general case of a non-scaling FFAG where the similarity of the orbits is no longer valid, the previous approach is repeated for a fixed Q value equivalent to $I = 0.5mA$ at injection. The results are shown in Figs. 5 and are in good agreement with Eq. (17).

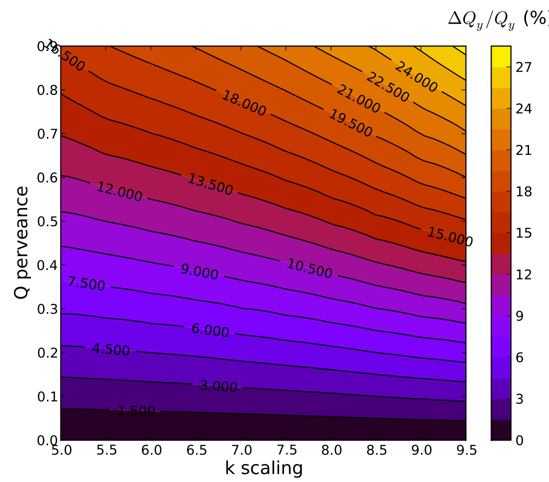
Now, back to the previous question: given that the space charge tune shift is larger at injection and decreases with the energy, for a scaling FFAG, the KV tune excursion is necessarily an increasing function of the energy in both planes. Therefore, the idea of the following method is to introduce a perturbation of the average field index of the magnets $(\delta k_F, \delta k_D)$ in such a way as to counteract the space charge effects, and thus produce a constant tune. As shown earlier, if $\kappa < 0$, this condition can be satisfied since the bare tune is a decreasing function of the energy in both planes. The problem writes in the following way:

$$\begin{cases} \nu_x = \nu_{x_0} + \delta\nu_x(\delta k_F, \delta k_D) + \delta\nu_x(\text{space charge}) \\ \nu_y = \nu_{y_0} + \delta\nu_y(\delta k_F, \delta k_D) + \delta\nu_y(\text{space charge}) \end{cases}$$

where all quantities are energy dependent. In order to determine $(\delta k_F, \delta k_D)$, one equates the RMS tune excursion of the KV beam $\nu_{x,y}^{rms}(sc)$ with the RMS tune excursion of the



(a) Horizontal plane.



(b) Vertical plane.

Figure 3: Contour plot of the space charge tune shift for a scaling FFAG as a function of the average field index k and the perveance Q (from tracking).

bare tunes. Recalling Eq. (10) for the latter, and solving for κ, this yields:

$$\kappa \in \left[\frac{\nu_x^{rms}(sc)}{a_x} : \frac{\nu_y^{rms}(sc)}{a_y}\right] \quad (19)$$

In order to test the previous scheme, we choose a lattice corresponding to $k_F = k_D = 7.6$. If we choose to maintain k_F fixed, then Eq. (19) yields $k_D \in [7.0 : 7.11]$. The results are shown in Fig.6 which proves the validity of this method.

Gaussian Beam

For a Gaussian beam distribution the situation is more complicated due to the large tune spread: while the betatron tune shift is the largest for the core particles, the large amplitude particles experience a near zero tune shift. Therefore, the previous approach of perturbing the field index of the magnets does not remediate the tune spread of the beam.

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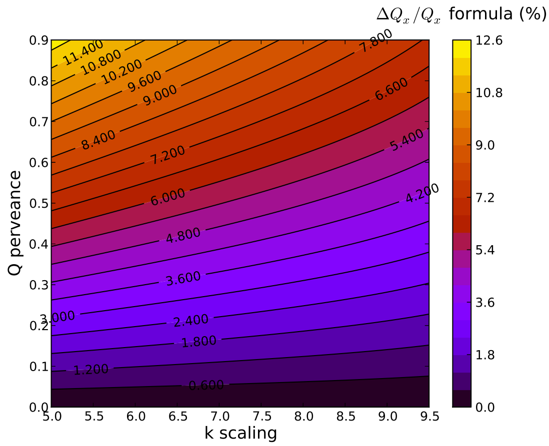
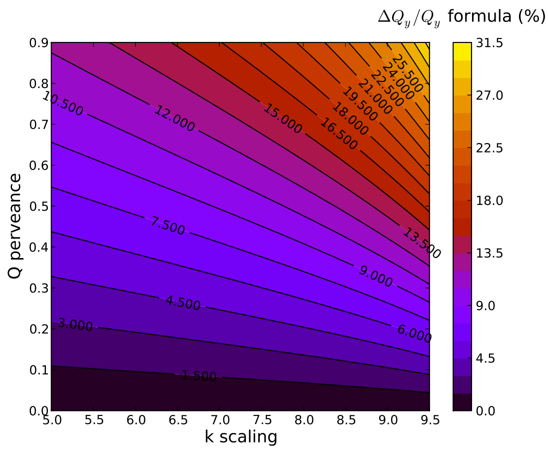


(a) Horizontal plane.



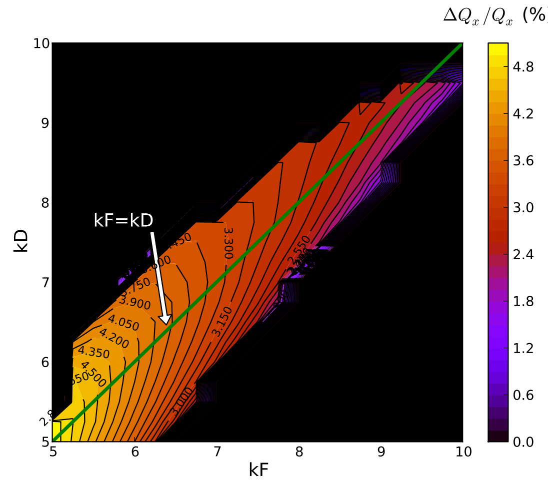
(b) Vertical plane.

Figure 4: Contour plot of the space charge tune shift for a scaling FFAG as a function of the average field index k and the perveance Q (analytical formula 18).

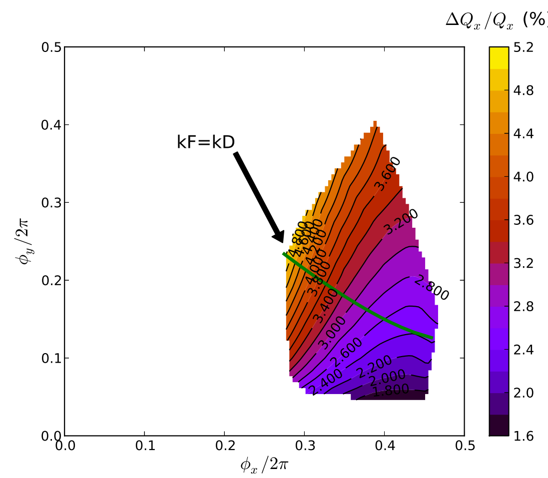
However, it remains valid for the core of the distribution which can be positioned in the tune diagram in a way to avoid harmful resonance crossings.

DISCUSSION AND PLANS

Presently, efforts are made to produce distributions with the linear space charge forces: uniform 3D ellipsoids are the only distributions whose internal force fields are linear functions of position [8]. Therefore, the ellipsoidal model is consistent with a KV distribution in the transverse phase space and a Neuffer distribution in the longitudinal phase space [9]. These two distributions are adequate approximations for the bunch modelling and can be correlated with other distributions via the concept of rms equivalent beams. So, the previous results of the KV beam model should hold in a more realistic case where the linear space charge forces are achieved. Future plans include the study of the resonance effects on the non-linear beam dynamics of FFAGs.



(a) (k_F, k_D) diagram.



(b) Phase advance diagram.

Figure 5: Contour plot of the horizontal space charge tune shift (a) for a non scaling FFAG as a function of the average field index (k_F, k_D) and the equivalent phase advance diagram (b).

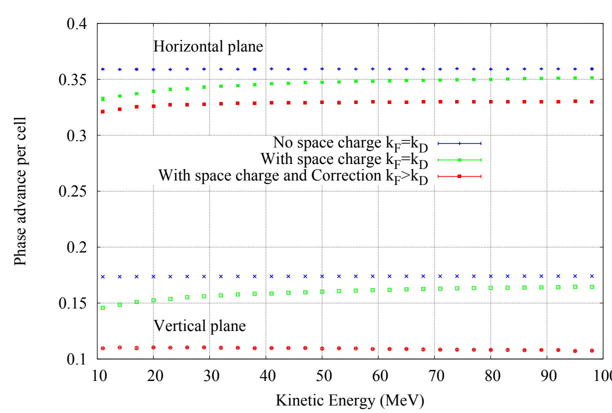


Figure 6: Phase advance per cell before and after perturbation of the average field index of the D-magnet: before correction $(k_F, k_D) = (7.6, 7.6)$ while after correction $(k_F, k_D) = (7.6, 7.11)$.

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REFERENCES

[1] S. L. Sheehy *et al.*, "CHARACTERISATION OF THE KURRI 150 MeV FFAG AND PLANS FOR HIGH INTENSITY EXPERIMENTS", presented at the 57th ICFA Advanced Beam Dynamics Workshop on High Intensity, High Brightness and High Power Hadron Beams (HB2014), Michigan, USA, November 2014, paper MOPAB27, pp 89–93.

[2] S. L. Sheehy *et al*, "Progress on simulation of fixed field alternating gradient accelerators", MOPJE077, IPAC'15, Virginia, USA, May 2015, paper MOPJE077, pp 495–498.

[3] M. Haj Tahar and F. Méot, "Transverse beam dynamics in nonlinear Fixed Field Alternating Gradient accelerators", BNL AP-Technote: C-A/AP/559, March 2016.

[4] K.R. Symon *et al.*, "Fixed-Field Alternating-Gradient Particle Accelerators", *Phys. Rev.*, vol. 103, no. 6, pp 1837–1859, Sep 1956.

[5] N.N. Bogoliubov and Y.A. Mitropolskii, "Asymptotic methods in the theory of nonlinear oscillations", Gordon and Breach, New York (1961).

[6] F. Méot, "Zgoubi Users Guide", Report CA/AP/470, BNL C-AD (2012).

[7] M. Haj Tahar *et al.*, "Space charge studies in FFAG using the tracking code ZGOUBI", MOPMN008, IPAC'15, Virginia, USA, May 2015, paper MOPMN008, pp 717–719.

[8] O. Luiten, S. Van der Geer, M. De Loos, F. Kiewiet, M. Van Der Wiel, Physical Review Letters, vol. 93, no. 9, p 094802, Aug 2004.

[9] M. Reiser, "Theory and Design of Charged Particle Beams", Wiley Series in Beam Physics and Accelerator Technology, Wiley-VCH, Berlin (2008).

ISBN 978-3-95450-178-6