Two Particle Model for Studying the Effects of Space-Charge Force on Strong Head-Tail Instabilities*

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Selected as an Editors' Suggestion

Outline

- Motivations of This Work
- Chao's Original Two Particle Model
- New Two Particle Model with Space-Charge
- Procedure to Identify Unstable Regions and to Compute Growth Rate
- Findings and Conclusions

Mysterious Simulation/Analytic Results

- During HB2014 Workshop, Kornilov and Blaskiewicz reported mysterious simulation and analytical results for beam instabilities with space-charge force.

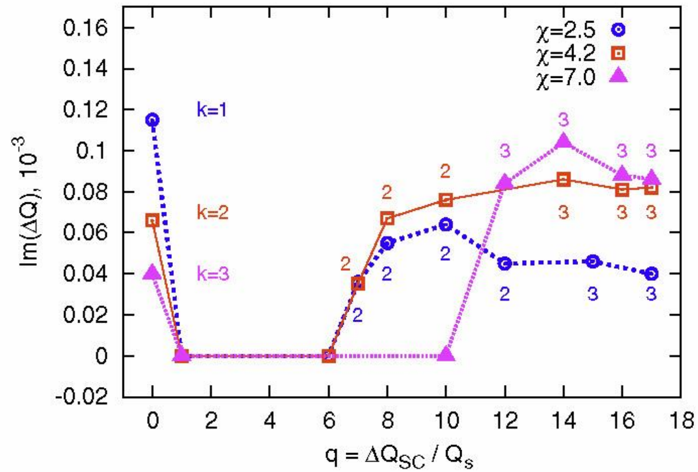


FIG. 10. Growth rates of the most unstable head-tail modes obtained in simulations for a Gaussian bunch for three different head-tail phases $\chi = \zeta\tau_b$ in a dependence on the space-charge parameter. The mode index k is given for each data point with the corresponding color.

V. Kornilov and O. Boine-Frankenheim
PRST-AB, 13, 114201 (2010)

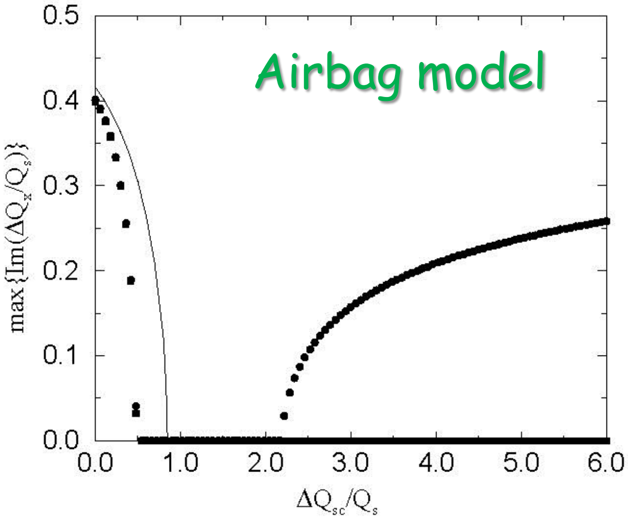


FIG. 1. Largest value of $\mathrm{Im}(\Delta Q_x/Q_s)$ as a function of $\Delta Q_{sc}/Q_s$ and $m_{\max}$ using matrix element (12). The value of W is twice the size needed to produce instability with $\Delta Q_{sc} = 0$: $m_{\max} = 1$, solid line; $m_{\max} = 5$, squares; $m_{\max} = 10$, circles.

M. Blaskiewicz
PRST-AB, 1, 044201 (1998)

Beam Instabilities with Space-Charge

- Many simulation results generally indicate that beam instability can be damped by a weak space-charge force, but the beam becomes unstable again when the space charge force is further increased.
- If the damping of beam instabilities is caused by the betatron tune spread (i.e., Landau damping) due to the non-linearity of the space-charge force,
  - A stronger space-charge force should be more effective in damping of beam instabilities.
- Why do many simulation results show the contrary?
- No definite answer to this question for the last ~20 years.

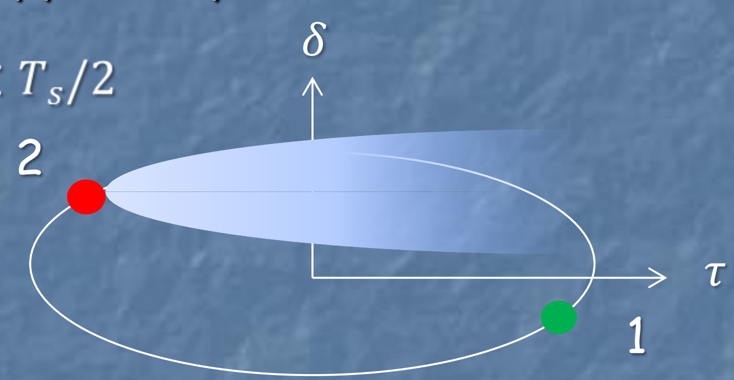
Invitation by Alex

- After the working session at HB2014, Alex has invited me and Mike to collaborate on study for effects of space-charge force on beam instabilities by modifying his famous two particle model for strong head-tail instabilities.
  - That was a fascinating idea.
    - We may be able to solve the mystery by using a simple model and mathematics for this complicated phenomenon.
    - We found later though that his proposed new two particle model did not work (a pity).
  - So, it turned out that the crux of the problem is to find a suitable new two particle model which is
    - A simple expansion of the original two particle model
    - Still analytically and exactly solvable.

Alex's Original Two Particle Model

- Let us first review the premise and treatment of Alex's original two particle model.
  - Two macro-particles executing synchrotron and betatron oscillations.
  - Their synchrotron oscillations have equal amplitude, but opposite phases.

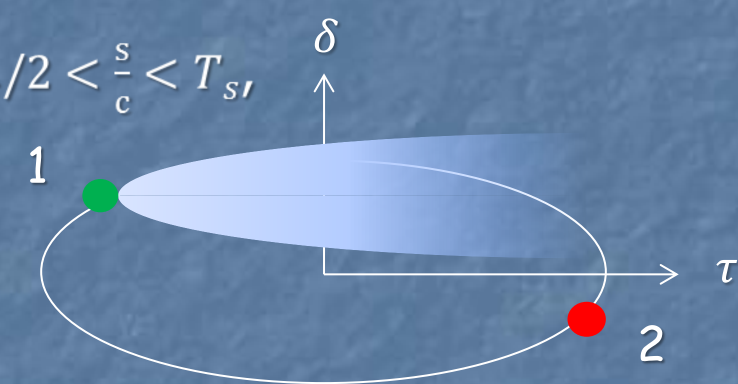
$0 < \frac{s}{c} < T_s/2$



$$y_1'' + \left(\frac{\omega_\beta}{c}\right)^2 y_1 = \frac{N r_0 W_0}{2\gamma C} y_2$$

$$y_2'' + \left(\frac{\omega_\beta}{c}\right)^2 y_2 = 0$$

$T_s/2 < \frac{s}{c} < T_s,$



$y_1 \rightleftarrows y_2$

Transfer Matrix

- Alex's two particle model is a peculiar system that
  - The oscillation of y_2 is a pure harmonic oscillator .
  - Thus, the amplitude of y_2 oscillation is a constant of motion.
- The motion of y_1 cannot be diagonalized, but its evolution can be found using the constant of motion of y_2:
- $\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}_{s=cT_s/2} = e^{-i\omega_\beta T_s/2} \begin{bmatrix} 1 & i\Upsilon \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}_{s=0}$ for $0 < \frac{s}{c} < T_s/2$
  - Here $\tilde{y}_2 = y_2 + i\frac{c}{\omega_\beta} y_2'$: amplitude with phase advance

$$\Upsilon = \frac{\pi N r_0 W_0 c^2}{4\gamma C \omega_\beta \omega_s}$$ ⟵ Dimensionless Wake Field Strength Parameter

Full Transfer Matrix

- The transfer matrix during the second half of the synchrotron oscillation period, $T_s/2 < \frac{s}{c} < T_s$, is obtained by exchanging the indices 1 and 2.
- Total Matrix

$$\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}_{s=cT_s} = e^{-i\omega_\beta T_s} \begin{bmatrix} 1 & 0 \\ i\Upsilon & 1 \end{bmatrix} \begin{bmatrix} 1 & i\Upsilon \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}_{s=0} =$$

$$e^{-i\omega_\beta T_s} \begin{bmatrix} 1 & i\Upsilon \\ i\Upsilon & 1-\Upsilon^2 \end{bmatrix} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}_{s=0}.$$

- Eigenvalues λ for the total matrix

$$\begin{bmatrix} 1 & i\Upsilon \\ i\Upsilon & 1-\Upsilon^2 \end{bmatrix} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Analysis of Eigenvalues

- Description of eigenvalues in Alex's book is geometrical (The eigenvalues are expressed by angles).

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As time evolves, the vector formed by the phasors $\tilde{y}_1$ and $\tilde{y}_2$ is repeatedly transformed by the 2×2 matrix in Eq. (4.43). Stability of the system is thus determined by the eigenvalues of this matrix. The two eigenvalues for the two modes (a + mode and a − mode) are

$$\lambda_{\pm} = e^{\pm i\phi}, \qquad \sin\frac{\phi}{2} = \frac{\Upsilon}{2}, \tag{4.44}$$

and the eigenvectors are

$$V_{\pm} = \begin{bmatrix} \pm e^{\pm i\phi/2} \\ 1 \end{bmatrix}. \tag{4.45}$$

Stability requires ϕ real, which is fulfilled if $|\sin(\phi/2)| \le 1$, or

$$\Upsilon \le 2. \tag{4.46}$$

For weak beams, $\Upsilon \ll 1$, we have $\phi \approx \Upsilon$. Near the instability, ϕ approaches π as Υ approaches 2.

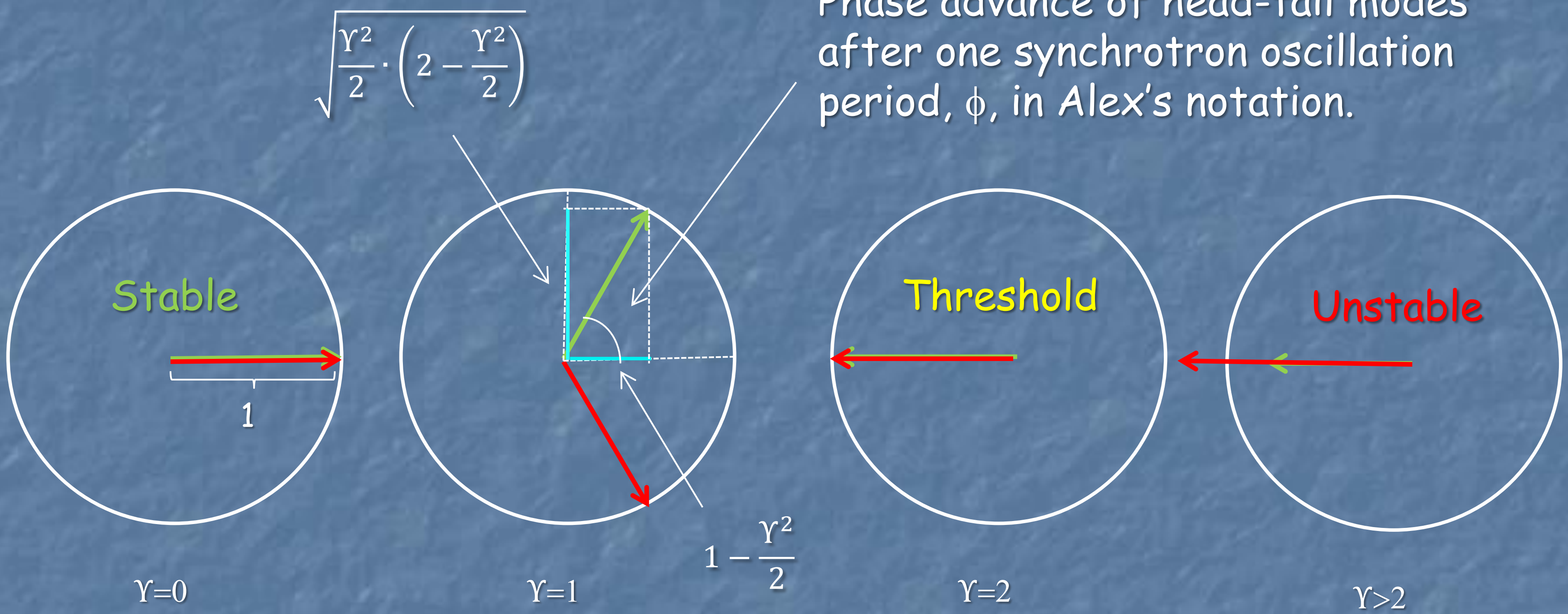
- Here, we try an algebraic and more intuitional expression.

Eigenvalues and Growth Rate

- The two eigenvalues are
  - $\lambda = \begin{cases} 1 - \frac{\Upsilon^2}{2} \pm \sqrt{\frac{\Upsilon^2}{2} \cdot \left(\frac{\Upsilon^2}{2} - 2\right)} & if\ \Upsilon^2 \geq 4 \\ 1 - \frac{\Upsilon^2}{2} \pm i\sqrt{\frac{\Upsilon^2}{2} \cdot \left(2 - \frac{\Upsilon^2}{2}\right)} & if\ \Upsilon^2 \leq 4 \end{cases}$
- At the threshold value of $\Upsilon^2 = 4$, the eigenvalue λ becomes exactly minus one ($\lambda = -1$) or
  - $\lambda = e^{\pm i\pi}$
- If $\Upsilon^2 \geq 4$, one of the solutions is unstable.
  - $\lambda = 1 - \frac{\Upsilon^2}{2} - \sqrt{\frac{\Upsilon^2}{2} \cdot \left(\frac{\Upsilon^2}{2} - 2\right)} \leq -1$

Instability Mechanism

Phase advance of head-tail modes after one synchrotron oscillation period, ϕ, in Alex's notation.



$$\lambda = 1-\frac{\Upsilon^2}{2}+i\sqrt{\frac{\Upsilon^2}{2}\cdot\left(2-\frac{\Upsilon^2}{2}\right)} \qquad \lambda = e^{+i\pi} \qquad \lambda = 1-\frac{\Upsilon^2}{2}+\sqrt{\frac{\Upsilon^2}{2}\cdot\left(\frac{\Upsilon^2}{2}-2\right)}$$

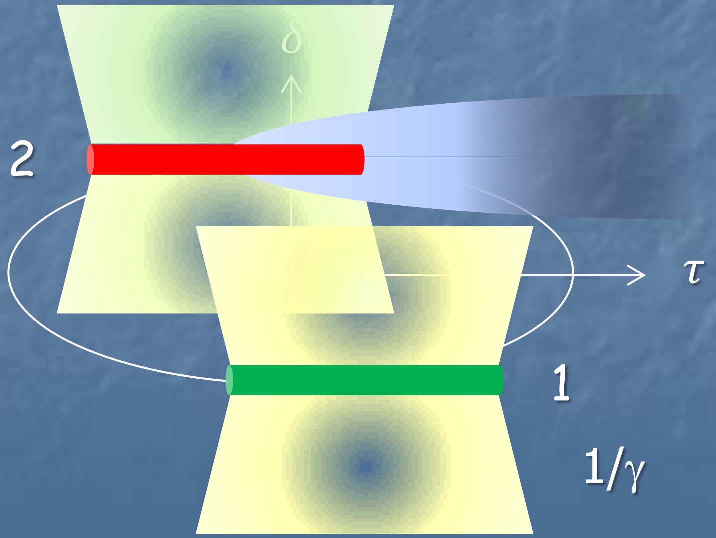
$$\lambda = 1-\frac{\Upsilon^2}{2}-i\sqrt{\frac{\Upsilon^2}{2}\cdot\left(2-\frac{\Upsilon^2}{2}\right)} \qquad \lambda = e^{-i\pi} \qquad \lambda = 1-\frac{\Upsilon^2}{2}-\sqrt{\frac{\Upsilon^2}{2}\cdot\left(\frac{\Upsilon^2}{2}-2\right)}$$

Transverse Mode-Coupling Instability

- It implies that the strong head-tail instability occurs by the mode coupling between the two solutions when the difference of their phase advances over one synchrotron period becomes exactly 2π.
- The growth rate g, when $\Upsilon^2 \geq 4$, is obtained by equating
  - $|\lambda| = e^{gT_s} = \sqrt{\frac{\Upsilon^2}{2} \cdot \left(\frac{\Upsilon^2}{2} - 2\right)} + \frac{\Upsilon^2}{2} - 1.$
- The growth factor $g \times T_s$ over one synchrotron oscillation:
  - $g \times T_s = \log\left\{\sqrt{\frac{\Upsilon^2}{2} \cdot \left(\frac{\Upsilon^2}{2} - 2\right)} + \frac{\Upsilon^2}{2} - 1\right\}$
    - It is an universal function only of the dimensionless wake field parameter Υ.

New Two Particle Model with Space Charge

- Two approximations:
  - Linear Model
    - The space-charge force is linear in the relative distance between the two particles.
  - Continuous Interaction Model
    - The two particles interact continuously and coherently with a space charge force in the transverse plane.



For $0 < \frac{s}{c} < T_s/2$

$$y_1'' + \left(\frac{\omega_\beta}{c}\right)^2 y_1 = K(y_1 - y_2) + W y_2$$

$$y_2'' + \left(\frac{\omega_\beta}{c}\right)^2 y_2 = K(y_2 - y_1)$$

$$W = \frac{N r_0 W_0}{2\gamma C} \qquad K = \frac{N r_0}{a^2 \beta^2 \gamma^3 C}$$

Mathematical Procedure

- Find new coordinates (eigenvectors) to diagonalize the system to two independent harmonic oscillators.
- Find constants of motion to describe those harmonic oscillators (they are often the amplitude of oscillations).
- Describe the system using the original coordinates and the constant of motions that we have just found.
- Calculate a transfer matrix for the amplitude of the particle motion (with the phase advance), not for the particle coordinates and its momentum.
  - This way, the matrix becomes 2x2, not 4x4.
- Find eigenvalues of a matrix for the full synchrotron oscillation period.

Weak Space-Charge Case (W≥K)

- By adding the space-charge term, the system has no trivial harmonic oscillator solution.
  - We can now apply the general eigenvalue technique.
- For given Υ (the dimensionless wake field parameter) and $\Delta\nu_{sc}/\nu_s$ (the dimensionless space-charge parameter),
  - $\mathrm{r} = \frac{K}{W} = \frac{\pi}{2\Upsilon}\left(\frac{\Delta\nu_{sc}}{\nu_s}\right) \leq 1$

$$\mathrm{y} = 2\sqrt{r(1-r)}$$

$$\tanh^2\left(\frac{\Upsilon}{2}\mathrm{y}\right) \leq \mathrm{y}^2$$

Yes → Stable

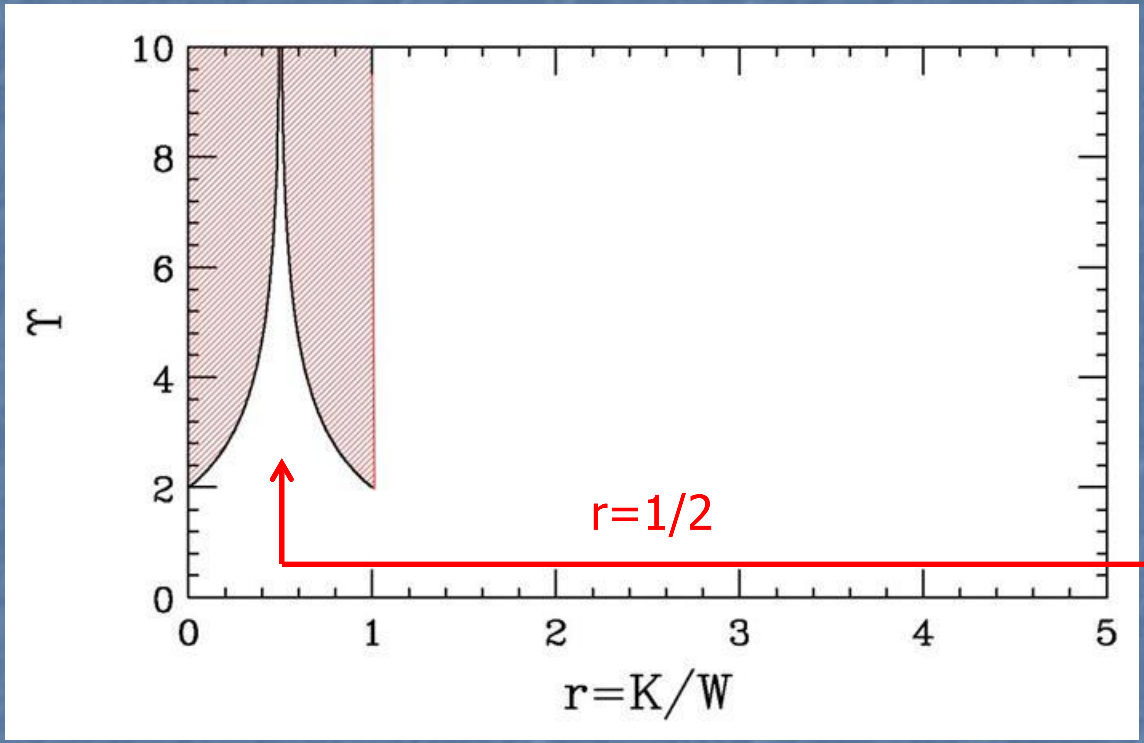
No → Unstable

Γ behaves like Υ

$$\frac{\Gamma^2}{2} = 2 \cdot \frac{1-y^2}{y^2} \cdot \frac{\tanh^2\left(\frac{\Upsilon}{2}\mathrm{y}\right)}{1-\tanh^2\left(\frac{\Upsilon}{2}\mathrm{y}\right)}.$$

Growth rate: $g = \frac{1}{T_s}\log\left\{\sqrt{\frac{\Gamma^2}{2}\cdot\left(\frac{\Gamma^2}{2}-2\right)} + \frac{\Gamma^2}{2} - 1\right\}$

Weak Space-Charge Case (W≥K)



The stability diagram for the weak space-charge case (r=K/W≤1).

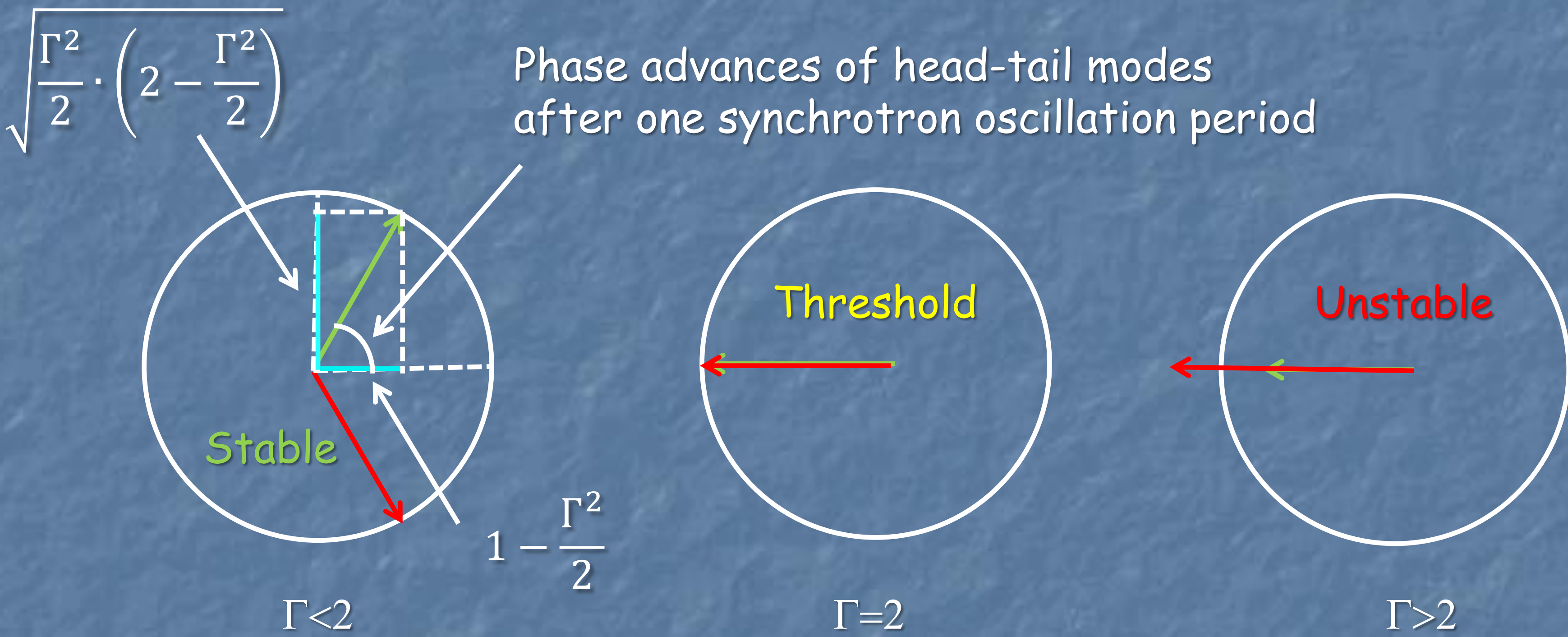
Unstable regions are shown shaded.

$$y_1'' + \left[\left(\frac{\omega_\beta}{c}\right)^2 - \frac{W}{2}\right] y_1 = \frac{W}{2} y_2$$
$$y_2'' + \left[\left(\frac{\omega_\beta}{c}\right)^2 - \frac{W}{2}\right] y_2 = -\frac{W}{2} y_1$$

Absolutely stable regardless of W

The white region between r=0 and 1 and above $\Upsilon = 2$ is a passband created by decoupling of the modes by the space-charge force.

Instability Mechanism



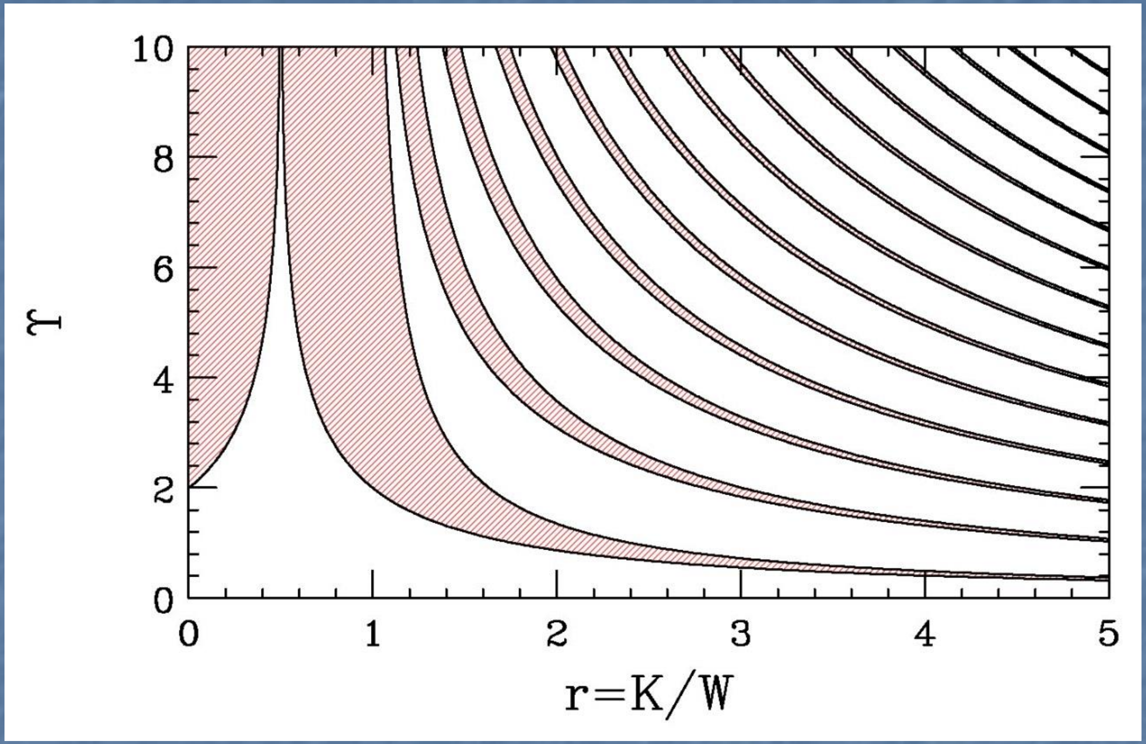
$$\lambda = 1-\frac{\Gamma^2}{2}\pm i\sqrt{\frac{\Gamma^2}{2}\cdot\left(2-\frac{\Gamma^2}{2}\right)}$$

$$\lambda = e^{\pm im\pi}$$

m=odd integer

$$\lambda = 1-\frac{\Gamma^2}{2}\pm\sqrt{\frac{\Gamma^2}{2}\cdot\left(\frac{\Gamma^2}{2}-2\right)}$$

Strong Space-Charge Case (K≥W)

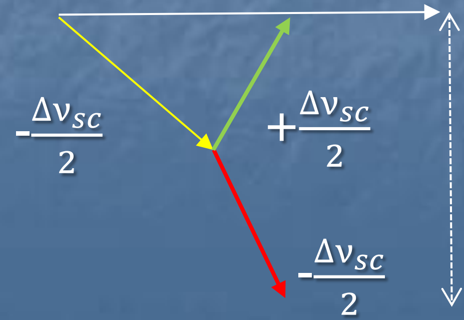


The stability diagram for the strong space-charge case ($r=K/W\geq1$).

The stability diagram for the weak space-charge case ($r=K/W\leq1$) is also plotted for completion.

Unstable regions are shown shaded.

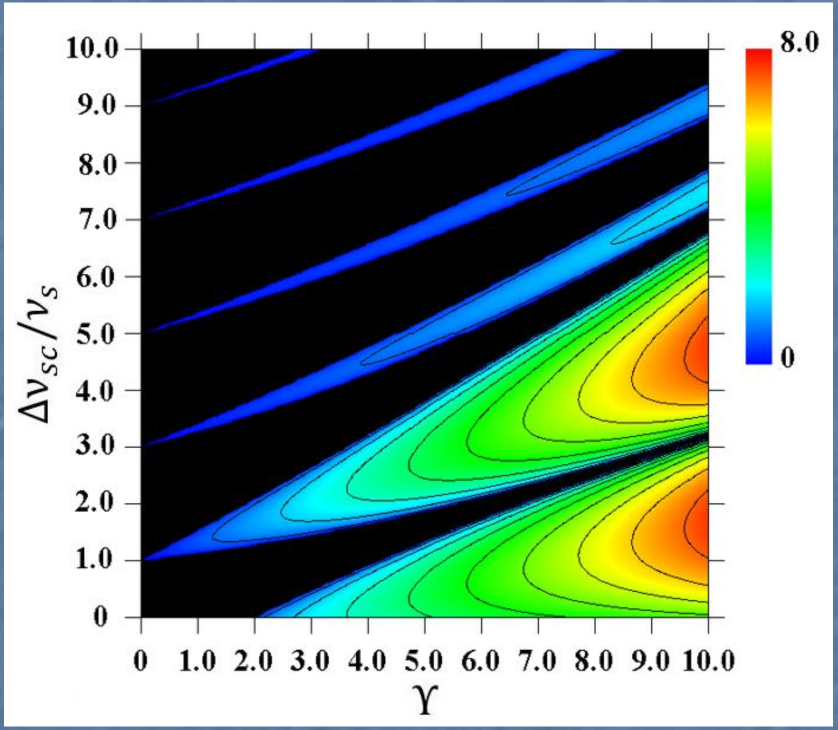
The mode-coupling condition is satisfied when $\Delta\nu_{sc}$ takes values around an odd integer times ν_s. Many stopbands appear.



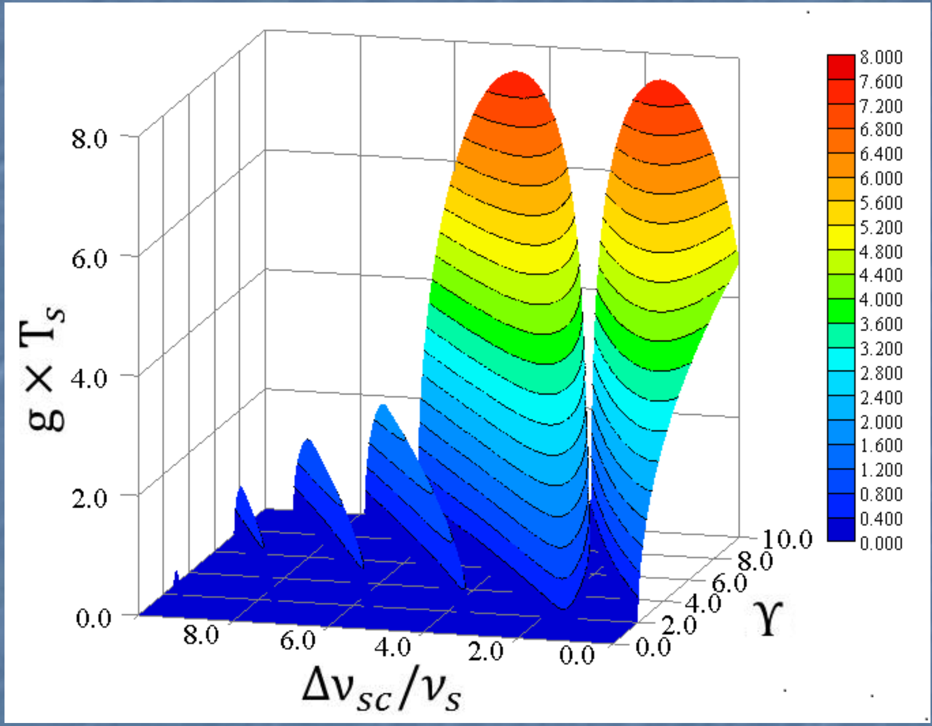
~ Odd integer times ν_s

Contour Plots for Growth Rate

These figures are all universal !



Flat contour plot for the growth factor $g \times T_s$ as a function of Υ and $\frac{\Delta\nu_{sc}}{\nu_s}$.



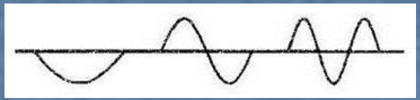
3-dimensional contour plot for the growth factor $g \times T_s$ as a function of Υ and $\frac{\Delta\nu_{sc}}{\nu_s}$.

Mode Coupling/Decoupling

- The damping of strong head-tail instabilities with a weak space-charge force is caused by decoupling of the modes due to additional tune shifts by the space-charge force, not the Landau damping due to the non-linearity of the space-charge force.
- As the space-charge force is increased, tune shifts by the space-charge force conversely restore the mode-coupling. But, a further increase of the space-charge force decuples the modes again.
- This mode coupling/decoupling behavior creates stopband structures as a function of the space-charge tune shift parameter and Υ.

Two Particles Play Many Modes

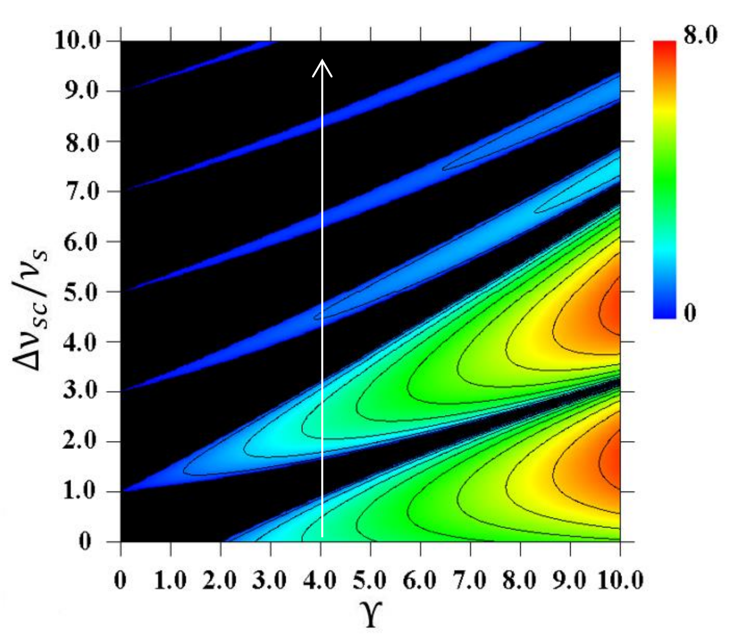
- In the present two particle model, there are only two modes in principle, but they play many different modes as if in a more general mode expansion method.
- Very roughly speaking, it appears that the one mode plays always the m=0 mode, while the other mode plays negative odd integer modes (m=-1,-3,-5,…) depending on the strength of the space-charge force.
- But, the two particles can have only one node at most in the oscillation envelop, while higher-order headtail modes would have multiple nodes.

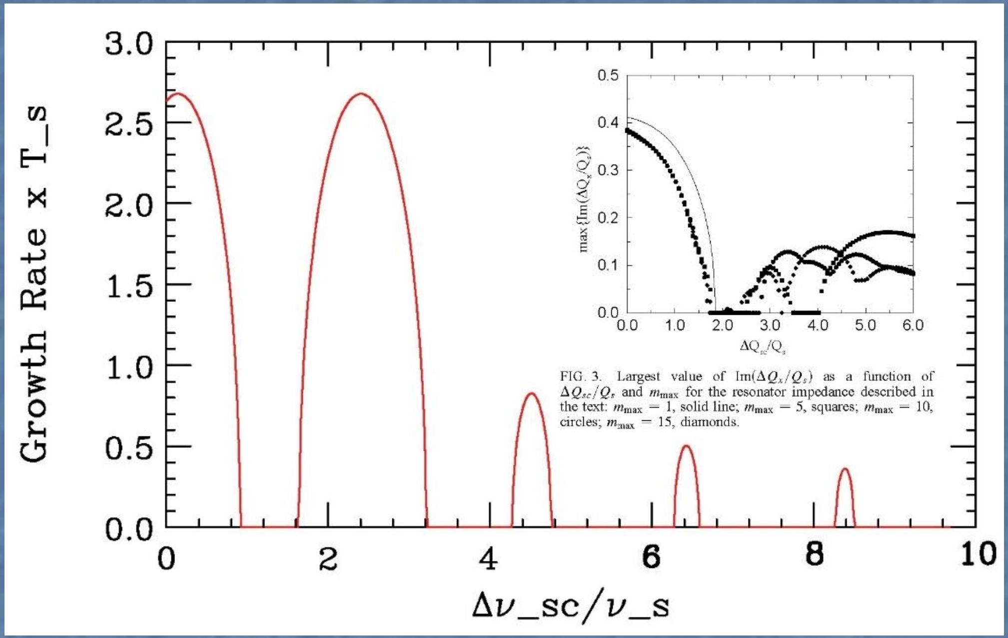


- In this sense, the present model may not depict an accurate picture of mode coupling between higher – order headtail modes at large space-charge tune shift.

Growth Rate as a Function of Space-Charge Tune Shift

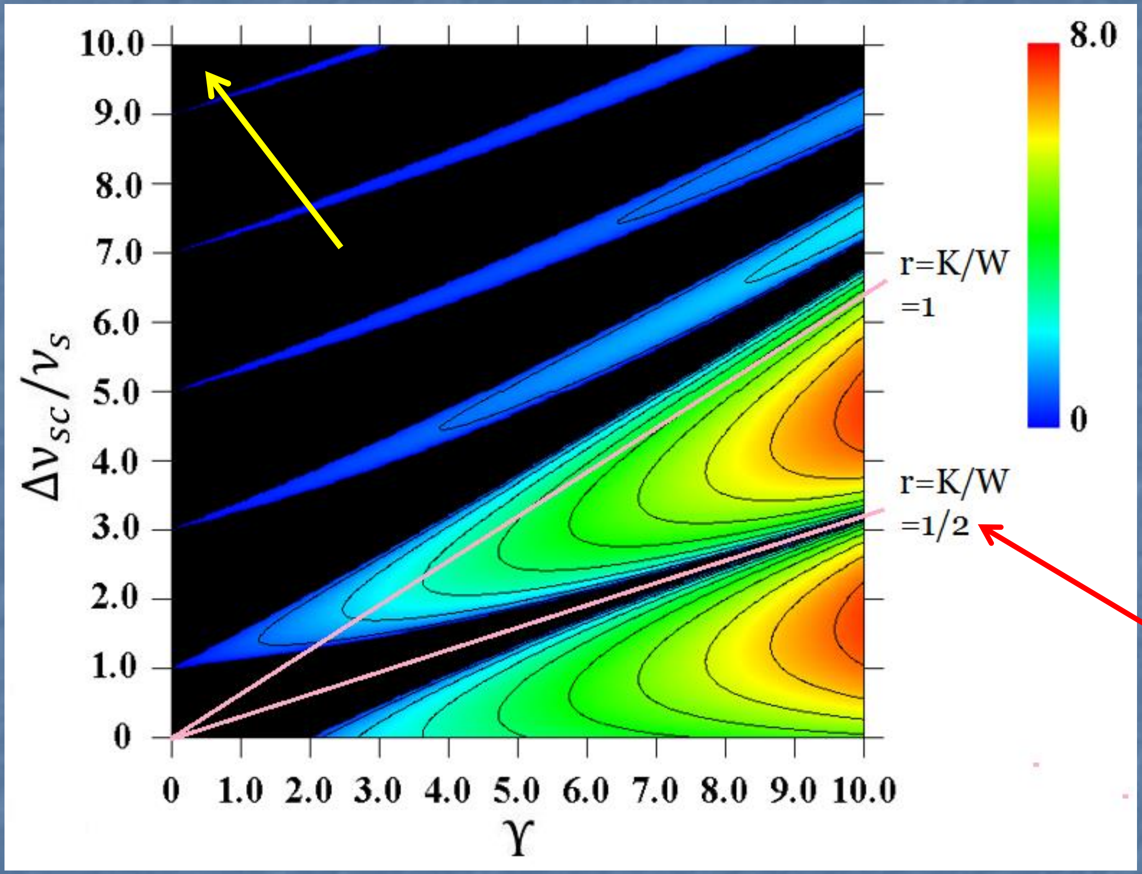
- $\Upsilon = 4$ case.





- It shows that the space-charge force loses its damping effect when it is too strong.
- It qualitatively reproduces typical behaviors shown in theoretical and simulation results.

Two Cases of Absolutely Stable Coupled Motions



As the space-charge force increases, Eqs. of motion approach to those for two pendulums connected with a spring.

$$y_1'' + \left(\frac{\omega_\beta}{c}\right)^2 y_1 = K(y_1 - y_2)$$
$$y_2'' + \left(\frac{\omega_\beta}{c}\right)^2 y_2 = K(y_2 - y_1)$$

Another absolutely stable motions.

$$y_1'' + \left[\left(\frac{\omega_\beta}{c}\right)^2 - \frac{W}{2}\right] y_1 = \frac{W}{2} y_2$$
$$y_2'' + \left[\left(\frac{\omega_\beta}{c}\right)^2 - \frac{W}{2}\right] y_2 = -\frac{W}{2} y_1$$

Why is the pure space-charge oscillation stable?

- Because there is no energy transfer or flow into the transverse oscillation externally or from the longitudinal motion of a beam.
  - In case of transverse head-tail beam instability, even if only transverse mode is excited in a structure, it has a longitudinal impedance and the head-tail mode gains energy from the longitudinal energy loss of the beam.
- Just like the two pendulum system connected with a spring, the space-charge force itself cannot excite beam instabilities:
  - It needs wake fields.
  - If the wake fields are sufficiently weaker than the space-charge force, the beam will stay stable.

Summary

- The present two particle model has no tune spread effect, since the space-charge force is linearized in the transverse position.
- However, the damping of beam instabilities with a weak space-charge force can be well explained by pure coherent kicks of the space-charge force:
  - They partially neutralize the coherent wake field kicks and decouple the modes.
- The damping by linear coherent kicks is unusual ?
  - No. To damp beam instabilities externally, we often use
    - Non-linear magnets such as octupoles for Landau damping by an incoherent tune spread.
    - Feedback system for linear coherent kicks to a beam.

Conclusions and Preview of Works to Come

- The purpose of the present simple model is, not to explain every effect of space-charge force on beam instabilities with numerical precision, but to suggest a simple picture of some of the essence of the physics of this complicated subject.
- We hope that it will provide a good starting point for young scientists to join this effort with their own models or improved versions of the present model so that the model becomes more physically accurate.
- We are currently working on
  - Effects of chromaticity on head-tail instabilities using the two particle mode and the Vlasov approach.
  - Inclusion of all three effects: wake fields, the chromaticity and the space-charge force.

“The Geography of Thought” by R. Nisbett
How Asians and Westerners Think Differently… and Why

- According to this book,
  - Westerners think that the World is simple and steady.
    - It is ruled by simple laws of nature and can be described by simple models.
    - They value principles.
  - Asians think that the World is complicated and rapidly changing.
    - It is too complicated even to describe.
    - There is no law of nature, since such a law is also changing all the time.
    - They value practicality.
  - That is why Westerners succeeded in creating and developing science called physics, while Asians failed.